1. Indices and Logarithms

1.1 Indices

Imagine you have a piece of paper. You tear it into two. You then proceed to tear both pieces again into another two. Now, you have 4 pieces of paper. If you tear it a third time, how many pieces would you have? What about 6 times? What about fifteen times?

Based on what you already know, tearing it a third time would give you:

2 x 2 x 2 = 8 pieces.

The sixth time would give you:

2 x 2 x 2 x 2 x 2 x 2 = 64 pieces.

Writing 2 x 2 x 2 x 2 ... endlessly is not only time consuming, but also a waste of space as well as impractical. This is where the indices come in. Let's define the index as follows:

If *a* is any number and *n* a positive integer (whole number) then the product of *a* with itself *n* times is called a raised to the power *n*, and written a^n ; i.e.,

$$a^n = \underbrace{a \times a \times ... \times a}_{\text{n times}}$$

where *a* is the base and *n* is the index.

Referring to the above example,

$$2^3 = 2 \times 2 \times 2 = 8 \qquad \text{base 2, index 3}$$
$$2^6 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64 \qquad \text{base 2, index 6}$$

1.1.1 Rules of Indices

For all positive values of *a* and all values of *m* and *n*, the following important rules apply to indices:

1. $a^m \times a^n = a^{m+n}$
2. $a^m \div a^n = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $a^1 = a$
5. $a^0 = 1$

Other properties of indices:

1. $a^m \times b^m = (ab)^m$
2. $\dfrac{a^m}{b^m} = \left(\dfrac{a}{b}\right)^m$

Examples:

a) $2^2 \times 2^3 = (2\times 2)\times(2\times 2\times 2) = 2^5 = 2^{2+3}$

b) $3^5 \div 3^2 = \dfrac{3\times 3\times 3\times 3\times 3}{3\times 3} = 3\times 3\times 3 = 3^3 = 3^{5-2}$

c) $(2^3)^2 = (2\times 2\times 2)\times(2\times 2\times 2) = 2^6 = 2^{3\times 2}$

d) From rule 2, $a^{n+1} \div a^n = a^{(n+1)-n} = a^1$

Also, $a^{n+1} \div a^n = \dfrac{a\times a\times ......a\times a}{a\times a.....\times a} = a$

$\therefore a^1 = a$

e) $a^n \times a^0 = a^{n+0} = a^n = a^n \times 1$

$\therefore a^0 = 1$

1.1.2 Negative Indices

Consider the following example.

$$a^2 \div a^5 = \frac{a\times a}{a\times a\times a\times a\times a} = \frac{1}{a\times a\times a} = \frac{1}{a^3}$$

By applying rule 2, $a^2 \div a^5 = a^{-3}$

Thus, we get $a^{-3} = \dfrac{1}{a^3}$

In general, we define the negative powers of numbers as:

$$a^{-n} = \frac{1}{a^n}$$

Examples:

a) $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

b) $5^2 \div 5^4 = 5^{(2-4)} = 5^{-2} = \frac{1}{5^2} = \frac{1}{25}$

1.1.3 Fractional Indices

If *a* is a positive number, then the square root of *a* is the number which multiplied by itself gives *a*. Thus, 3 is the square root of 9 since $3^2 = 9$. We write $3 = \sqrt{9}$. Note that, by definition, $\sqrt{a} \times \sqrt{a} = a$. This gives us a way of interpreting $a^{\frac{1}{2}}$.

As rule 1 is true for all values of *m* and *n*,

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\left(\frac{1}{2}+\frac{1}{2}\right)} = a^1 = a = \sqrt{a} \times \sqrt{a}$$

$$\therefore a^{\frac{1}{2}} = \sqrt{a}$$

The general rule is that, if *a* is a positive number and *n* is a positive integer, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

where $\sqrt[n]{a}$ is the *n*-th root of *a*.

Note: $\underbrace{a^{\frac{1}{n}} \times a^{\frac{1}{n}} \ldots\ldots\ldots\ldots \times a^{\frac{1}{n}} \times a^{\frac{1}{n}}}_{n\text{-times}} = (a^{\frac{1}{n}})^n = a^{\frac{1}{n}\times n} = a$

Now consider $a^{\frac{p}{q}}$.

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}\times 2}$$

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}\times 3}$$

By proceeding in this way q times;

$$\underbrace{a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \ldots\ldots\ldots \times a^{\frac{p}{q}} \times a^{\frac{p}{q}}}_{q\text{-times}} = a^{\frac{p}{q} \times q} = a^{p}$$

Therefore, $a^{\frac{p}{q}}$ is the q-th root of a^p.

i.e., $a^{\frac{p}{q}} = \sqrt[q]{a^p}$

Note: $a^{\frac{p}{q}} = a^{\frac{1}{q} \times p} = (a^{\frac{1}{q}})^p$

Examples:

1) $100^{\frac{1}{2}} = \sqrt{100} = 10$
2) $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$
3) $27^{\frac{5}{3}} = (27^{\frac{1}{3}})^5 = 3^5 = 243$

1.2 Logarithms

Let *a* and *y* be positive real numbers and let $y = a^x$: Then *x* is called the logarithm of *y* to the base *a*. We write this as

$$x = \log_a y$$

Examples:

1. Since $8 = 2^3$, then $3 = \log_2 8$
2. Since $81 = 3^4$, then $4 = \log_3 81$
3. Since $2 = \sqrt{4} = 4^{\frac{1}{2}}$, then $1/2 = \log_4 2$
4. Since $5^{-1} = \frac{1}{5}$, then $-1 = \log_5 (\frac{1}{5})$

1.2.1 Rules of Logarithms

Let *a*, X, Y be positive real numbers and k be any number. Then the following rules apply to logarithms.

1. $\log_a xy = \log_a x + \log_a y$
2. $\log_a (x/y) = \log_a x - \log_a y$
3. $\log_a (x^k) = k \log_a x$
4. $\log_a a = 1$
5. $\log_a 1 = 0$

1. **Logarithm of Product**

$$\log_a XY = \log_a X + \log_a Y$$

Examples:

a) $\log_4 2 + \log_4 8 = \log_4 (2 \times 8) = \log_4 16 = 2$
b) $\log_5 20 + \log_5 (\frac{1}{4}) = \log_5 (20 \times \frac{1}{4}) = \log_5 5 = 1$
c) $\log_3 45 - \log_3 5 = \log_3 (\frac{45}{5}) = \log_3 9 = 2$

2. **Logarithm of Quotient**

$$\log_a (X \setminus Y) = \log_a X - \log_a Y$$

Examples:

a) $\log_2 40 - \log_2 5 = \log_2 \frac{40}{5} = \log_2 8 = 3$
b) If $\log_3 5 = 1.465$ then we can find $\log_3 0.6$.
As 3/5=0.6, $\log_3 0.6 = \log_3 (3/5) = \log_3 3 - \log_3 5$
As $\log_3 3 = 1$ from rule 4, $\log_3 5 = 1 - 1.465 = -0.465$

3. **Logarithm of a Power**

$$\log_a(X^k) = k\log_a X$$

Examples:

a) Find $\log_{10}(1/100000)$.
$\log_{10}(1/100000) = \log_{10}(1/10^5) = \log_{10}(10^{-5}) = -5\log_{10}10$
As $\log_{10}10 = 1$, $\log_{10}(1/100000) = -5$

b) Find $\log_{81}9$.
As $\sqrt{81} = 9$, $\log_{81}9 = \log_{81}\sqrt{81} = \log_{81}(81)^{1/2} = \frac{1}{2}\log_{81}81$
As $\log_{81}81 = 1, \log_{81}9 = \frac{1}{2}$

1.2.2 Change of Bases

While the most commonly used base in finding logarithms is base 10, finding the logarithms to other bases are also required. The following rule relates logarithms in one base to logarithms in a different base.

$$\log_a c = \log_a b \times \log_b c$$

Proof:

Let x=$\log_a b$ and y=$\log_b c$
Then by the definition of logarithms, $b = a^x$ and $c = b^y$

i.e., $c = b^y = (a^x)^y = a^{xy}$

by the definition of logarithms,

$$\begin{aligned}\log_a c &= \log_a a^{xy}\\ &= xy\\ &= \log_a b \times \log_b c\end{aligned}$$

Examples:

a) $\log_{10}3 = 0.47712$ and $\log_{10}7 = 0.84510$. Find $\log_3 7$.

By rearranging the above rule, we have

$$\log_b c = \frac{\log_a c}{\log_a b}$$

So, $\log_3 7 = \dfrac{\log_{10} 7}{\log_{10} 3} = \dfrac{0.84510}{0.47712} = 1.77124$

b) $\log_{10} 5 = 0.69897$. Find $\log_2 5$.

Since 2=10/5,

$$\begin{aligned}\log_{10} 2 &= \log_{10}(10/5)\\ &= \log_{10} 10 - \log_{10} 5\\ &= 1 - 0.69897\\ &= 0.30103\end{aligned}$$

Then,

$$\log_2 5 = \frac{\log_{10} 5}{\log_{10} 2} = \frac{0.69897}{0.30103} = 2.32193$$

1.2.3 Graph of exponential and logarithmic functions

1.2.3 Graph of exponential functions

An exponential function is any function that can be written in the form

$$y = a^x$$

where a is a positive real number other than 1.

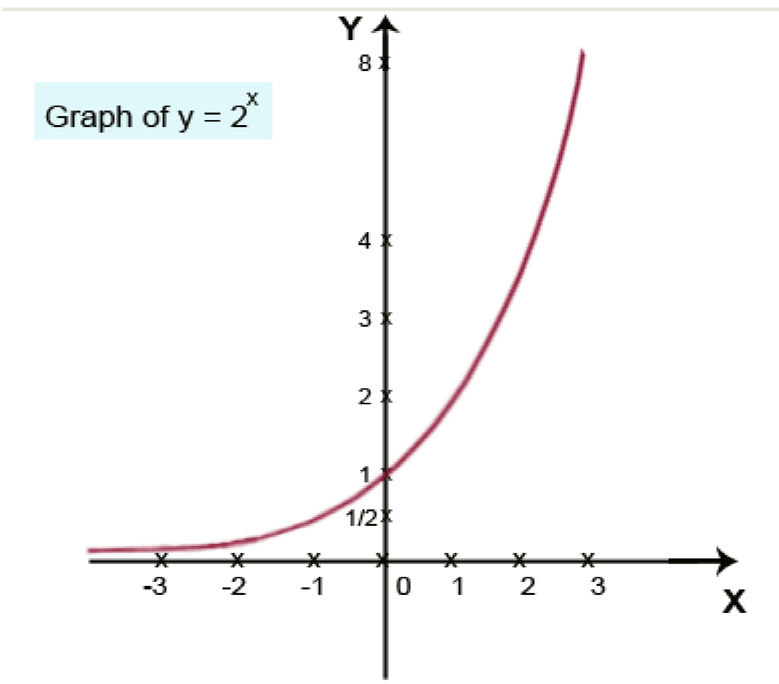
eg., $y = 2^x$, $y = 3^x$, $y = (\frac{1}{3})^x$

Let's find some values for the exponential function $y = 2^x$ for some values of x.

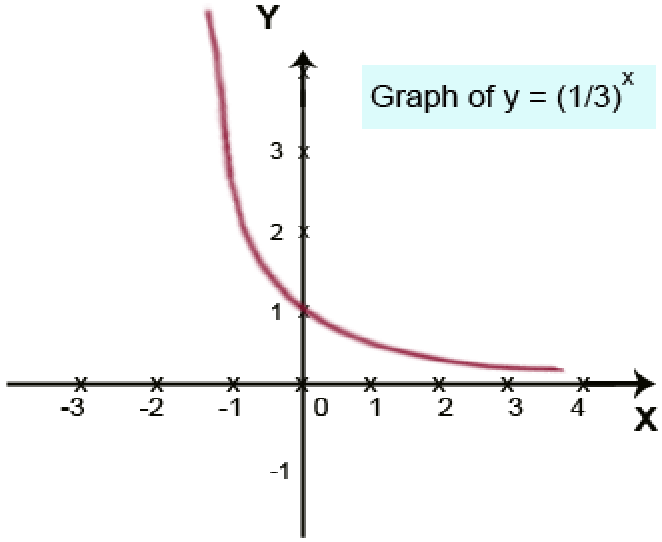
x	2^x	y
-3	2^{-3}	1/8
-2	2^{-2}	¼
-1	2^{-1}	½
0	2^0	1

1	2^1	2
2	2^2	4
3	2^3	8

Let's now sketch the graph of the exponential function $y = 2^x$ using the above (x,y) coordinates. Marking these (x,y) points on the graph paper and connecting them with a smooth curve, we have the graph of $y = 2^x$ shown in the following figure.



Notice that the graph approaches x-axis without actually intersecting the x-axis. In order for the graph of $y = 2^x$ to intersect the x-axis, there must be a value of x such that $0 = 2^x$. As no such value of x exists, the graph of $y = 2^x$ cannot intersect the x-axis. Note that when x increases, y also increases. But in the graph of $y = (\frac{1}{3})^x$, y decreases when x increases.



1.2.3.2 Graph of logarithmic functions

An logarithmic function is any function that can be written in the form

$$y = \log_a x$$

where a is a positive real number other than 1.

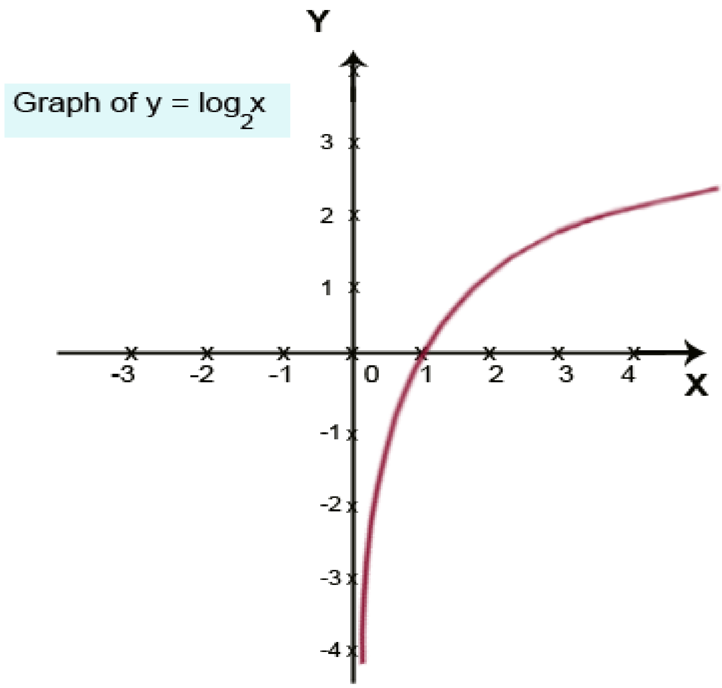
eg. $y = \log_2 x$

The equation $y = \log_2 x$ is, by definition, equivalent to the exponential equation

$$x = 2^y$$

Therefore, we can consider logarithmic functions as exponential functions.

The graph of $y = 2^x$ was given in the previous subsection. If we simply exchange the x and y in $y = 2^x$, we will get $x = 2^y$. Therefore, we say the logarithmic function is the inverse of an exponential function. Hence, the graph of $x = 2^y$ can be drawn as in the following figure, which is also the graph of $y = \log_2 x$.



Note: Simply reflect the graph of $y = 2^x$ about the line $y = x$ to get the graph of $x = 2^y$.