IT2103

Mathematics for Computing 1

Logic

Predicates and Quantifiers

Predicates

- **“x > 10, where x is an integer”**

- **Is this statement proposition?**

- (A **Proposition** is a sentence or statement that can be true or false)

Predicates

- **“x > 10, where x is an integer”**

- This sentence is **not a proposition** as its **truth value cannot be determined**.

- This is due to the presence of a variable x.

Predicates

- **"x > 10, where x is an integer"**

- Once a specific value is assigned to the variable x, then this sentence may become specifically true or false.
  - **If x = 1: False**
  - **If x = 26: True**

Such sentences are called **Predicates**.

Predicate Variables

- A Predicate may contain one or more **variables**.
- We can represent a predicate as a function.
  - For example, **“x > 10, where x is an integer”** be represented as **P(x)**

Example

Predicate Variables

- Consider **$x < y$ where $x, y \in R$**
- This predicate has **two** variables

- Consider **$x > 10$ where $x \in N$**
- This predicate has **one** variable

Definition
Universe of Discourse

- The set of values that a variable (or variables) in a predicate can take is called the **Universe of Discourse** (or the **Domain**) of that variable (or those variable).

Example
Universe of Discourse

- Consider **$x < y$ where $x, y \in R$**

- The **Universe of Discourse or Domain** of both variables x and y is the set of real numbers, R.

Definition
Truth Sets

- Let P(x) be a predicate and A be the Domain of x.
- Then, P(x) could be:
  - true for all values of A,
  - true for some values of A
  - true for no values of A.
- The set of all elements in A for which P(x) is true is called the **truth set** of the predicate P(x).

Example

Truth Sets

- Consider **P(x) : x + 5 > 10, x ∈ N**

- Let x ∈ N
- P(x) is True is and only if
  - x + 5 > 10
  - That is, x > 5
  - That is, x = 6 or 7 or 8 or …

- Truth set is **{6, 7, 8,…}**

The **Truth Set** of a predicate is always a subset of its **Domain**

Example
Truth Sets

- Consider **P(x) : x + y = 4 , x ∈ N, y ∈ N**

- By inspection <u>P(x) is true if and only if</u>
  - (x = 1 and y = 3) or
  - (x = 2 and y = 2) or
  - (x = 3 and y = 1)

- Truth set is **{(1, 3), (2, 2), (3, 1)}**

(x, y) ordered pairs

Universal Quantifier

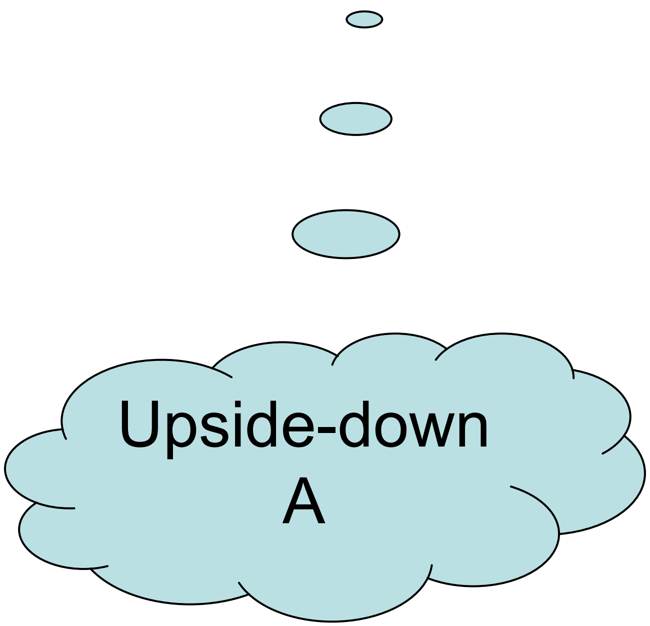
- Consider $\mathbf{x^2 \geq 0}$ **where** $\mathbf{x \in R}$**.**
- We know that $x^2 \geq 0$ is **true for any given value of x in R**.
- We can say:
- "For all $x \in R$, $x^2 \geq 0$ is true"

Definition

Universal Quantifier $\forall$

- We can denote “For all” by the symbol “$\forall$”. The symbol $\forall$ is called the **universal quantifier** and reads as “for any”, “for every” or “ for all”

$\forall$ **$\forall$ x $\in$ R, $x^2 \geq 0$** can be read as:
- “For any $x \in R$, $x^2 \geq 0$ is true”
- “For every $x \in R$, $x^2 \geq 0$ is true”
- “For all $x \in R$, $x^2 \geq 0$ is true”



Example

Universal Quantifier

- Consider P(x) is $\mathbf{x \geq 1}$

  $\forall x \in N$, P(x) is True

- $\forall x \in R$, P(x) is False
  Counter example:
  P(0.5) is false because $0.5 < 1$

Example
Universal Quantifier

$\forall$**x**$\in$**N, **$\forall$**y**$\in$**N, x+y > 1**

"For all x$\in$N, for all y$\in$N, x + y > 1"
This is True.

We can extend this to having any number of quantifiers:
$\forall x \in N$, $\forall y \in N$, $\forall z \in N$, $\forall w \in N$…

Existential Quantifier

Consider **$x + 7 = 10$ where $x \in N$**.

Since, $3 + 7 = 10$, this statement is **true for at least one value of x**.

We can say:
“There exists some $x \in N$ such that $x + 7 = 10$”

Definition

Existential Quantifier ∃

- We can denote “There exists” by the symbol “∃”. The symbol ∃ is called the existential quantifier and reads as “There exists”.

$\forall$ **$\exists x \in N, x + 7 = 10$** can be read as:

– “There exists $x \in N$ such that $x + 7 = 10$ is true”

Mirror-image of E

Example
Existential Quantifier

- Consider **P(x) is x + 23 = 25.5**

- $\exists x \in R$, P(x) is True
- P(2.5) is True because 2.5 + 23 = 25.5

- $\exists x \in N$, P(x) is False

Example
Existential Quantifier

- **$\exists x \in N, \exists y \in N, x > y$**

- "There exists some x $\in$N and there exists some y $\in$N such that x>y"

- This is True. For example, 2>1,

Quantifier Negation

$\forall$ x, P(x)
"For all x, P(x) is true"

not ($\forall$ x, P(x))
"not (For all x, P(x) is true)"

"There exists some x such that P(x) is false."
$\exists$x, ~P(x)

Quantifier
Negation

$\exists$ x, P(x)
"There exists some x such that P(x) is true"

not ($\exists$ x, P(x))
"not (There exists some x such that P(x) is true)"

"For all x, P(x) is false."
$\forall$x, ~P(x)

Example

Negation

- $\forall x \in N, x > 100$

  $\exists x, x \leq 100$

Quantifiers change

$\forall$ Becomes $\exists$

$\exists$ Becomes $\forall$

$\exists x \in N, x + 2 = 5$

$\forall x \in N, x + 2 \neq 5$

Predicate is negated

Mixed Quantifiers

- "**For all** $x \in N$, **there exists** some $y \in N$ such that $x \leq y$."
- "Given any natural number x, we can find another natural number y, such that x is less than y"
- We can write this as:
  $\forall x \in N, \exists y \in N, x < y.$

Mixed Quantifiers

- $\forall x \in N, \exists y \in N, x < y$

- Let $x \in N$
- Let $y = x+1$.
- Then $y \in N$ and $x < y$.

- Therefore, "$\forall x \in N, \exists y \in N, x < y$" is true.

Mixed Quantifiers
Order

- $\forall x \in N, \exists y \in N, x < y$
- $\exists x \in N, \forall y \in N, x < y$

- Are these both the same?

Mixed Quantifiers
Order

- **$\forall x \in N, \exists y \in N, x < y$**
- “For all $x \in N$, there exists some $y \in N$ such that $x < y$.”
  - **This is True**

- **$\exists x \in N, \forall y \in N, x < y$**
- “There exists some $x \in N$ such that for all $y \in N$, $x < y$.”

Mixed Quantifiers
Order

- $\exists x \in N, \forall y \in N, x < y$ - We prove this **False** by contradiction.

- <u>Suppose $\exists x \in N, \forall y \in N, x < y$ is true</u>
- Then for some constant $\mathbf{x_0} \in N, \forall y \in N, x_0 < y$
- Since, $1 \in N$, $x_0 < 1$ must be true
- But, all natural numbers (N) are greater or equal to one!
- This is a Contradiction!!!

- Hence, "$\exists x \in N, \forall y \in N, x < y$" must be false

Mixed Quantifiers
Order

- $\forall$**x**$\in$**N, **$\exists$**y**$\in$**N, x < y**
- "For all x$\in$N, there exists some y$\in$N such that x < y."

  – **True**

- $\exists$**x**$\in$**N, **$\forall$**y**$\in$**N, x < y**
- "There exists some x$\in$N such that for all y$\in$N, x < y."

  - **False**

Mixed Quantifiers

Negation

- $\forall x \in N, \exists y \in N, x < y$
  - not $(\forall x \in N, \exists y \in N, x < y)$
  - $\exists x \in N$, not$(\exists y \in N, x < y)$
  - $\exists \mathbf{x} \in \mathbf{N}, \forall \mathbf{y} \in \mathbf{N}, \mathbf{x \geq y}$
- $\exists x \in N, \forall y \in N, x < y$
  - $\forall \mathbf{x} \in \mathbf{N}, \exists \mathbf{y} \in \mathbf{N}, \mathbf{x \geq y}$

Quantifiers change
$\forall$ Becomes $\exists$
$\exists$ Becomes $\forall$

Predicate is negated

Thank you!!!