Techniques of Counting

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Learning Outcomes

After completing this module students will be able to

- Calculate the number of elements in certain mathematically defined sets where ordinary methods of counting are tedious
- Calculate number of possible outcomes of elementary combinatorial processes such as permutations and combinations

Permutation

A permutation is an arrangement of objects in different orders.

The order of the arrangement is important!! For example, the number of different ways 3 students can enter school can be shown as 3!, or $3 \cdot 2 \cdot 1$, or 6. There are six different arrangements, or permutations, of the three students in which all three of them enter school.

The notation for a permutation: ${}_nP_r$

n is the **total** number of objects

r is the number of objects chosen (want)

(Note if $n = r$ then ${}_nP_r = n!$)

Permutation

- To find the permutations of {A, B, C, D, E}, there are:
  - Five possible choices for the first item
  - Four possible choices for the second item
  - Three remaining possible choices for the third item
  - Two remaining possible choices for the fourth item, and
  - Only one possible "choice" for the final item
- For any positive integer N, we define N! ("N factorial") as the product of all the positive integers up to and including N
  - Example: 5! = 1 * 2 * 3 * 4 * 5 = 120
- Given any N *distinct* items, there are N! possible permutations of those items

Permutation

By the rule of product,

The number of permutations of n things taken r at a Time

$$P(n,r) = n(n - 1)(n - 2) \ldots (n - r + 1)$$

Note:

$$P(n,r) = \frac{n!}{(n-r)!}$$

Permutations

with Specific Arrangements

EXAMPLE:

Use the letters in the word " square " and tell how many 6-letter arrangements, with no repetitions, are possible if the :

a) first letter is a vowel.

b) vowels and consonants alternate, beginning with a consonant.

Solution:

Part a:

When working with arrangements, I put lines down to represent chairs.

Before starting a problem I decide how many chairs I need to fill and then work from there

I need six "chairs" (6-letter arrangements)

______ ______ ______ ______ ______ ______

The first of the six chairs must be a vowel (u ,a , e). There are three ways to fill the first chair.

$\underline{\ \ 3\ \ } \cdot \_\_\_\_\_\_ \ \_\_\_\_\_\_ \ \_\_\_\_\_\_ \ \_\_\_\_\_\_ \ \_\_\_\_\_\_$

After the vowel has been placed in the first chair, there are 5 letters left to be arranged in the remaining five chairs.

$\underline{\ \ 3\ \ } \cdot \underline{\ \ 5\ \ } \cdot \underline{\ \ 4\ \ } \cdot \underline{\ \ 3\ \ } \cdot \underline{\ \ 2\ \ } \cdot \underline{\ \ 1\ \ }$ or

$$3 \cdot {}_5P_5 = 3 \cdot 120 = 360$$

Solution:

Part b:

I need six "chairs" (6-letter arrangements)

______ ______ ______ ______ ______ ______

Beginning with a consonant, every other chair must be filled with a consonant. (s ,q , r)

$\underline{\quad 3 \quad} \cdot \underline{\qquad} \cdot \underline{\quad 2 \quad} \cdot \underline{\qquad} \cdot \underline{\quad 1 \quad} \cdot \underline{\qquad}$

The remaining chairs have the three vowels to be arranged in them:

$\underline{\quad 3 \quad} \cdot \underline{\quad 3 \quad} \cdot \underline{\quad 2 \quad} \cdot \underline{\quad 2 \quad} \cdot \underline{\quad 1 \quad} \cdot \underline{\quad 1 \quad}$

$= 36$

Permutations

with Repetition

In general, repetitions are taken care of by dividing the permutation by the number of objects that are identical!

Example1:

1. How many different 5-letter words can be formed from the word APPLE ?

$$\frac{{}_5P_5}{2!} = \frac{5\cdot4\cdot3\cdot2\cdot1}{2\cdot1} = \frac{120}{2} = 60\,words$$

You divide by 2! Because the letter P repeats twice

Permutations

with Repetition

Example:

2. How many different six-digit numerals can be written using all of the following six digits:

4,4,5,5,5,7

$$\frac{{}_6P_6}{2!3!} = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1\quad\cdot\quad 3\cdot 2\cdot 1} = \frac{720}{12} = 60$$

Two 4s repeat and three 5s repeat

Permutations

with Repetition

The number of different permutations of *n* objects, where there are *n1* indistinguishable objects of type 1, *n2* indistinguishable objects of type 2, …, and *nk* indistinguishable objects of type k, is

$$\frac{n!}{n_1!n_2!..n_k!}$$

Combinations

A **combination** is a set of objects in which order is *not* important.

The number of combinations of n things taken r at a time

$$C(n,r)=\binom{n}{r}=\frac{P(n,r)}{r!}=\frac{n!}{(n-r)!\,r!}$$

or

$${}_nC_r=\frac{{}_nP_r}{r!}$$

Combinations

Example1: Evaluate $7C2 = \frac{7 \cdot 6}{2 \cdot 1} = \frac{42}{2} = 21$

Example2: There are 12 boys and 14 girls in Mrs.Schultzkie's math class. Find the number of ways Mrs. Schultzkie can select a team of 3 students from the class to work on a group project. The team consists of 1 girl and 2 boys.

boy $_{12}C_2$ girl $_{14}C_1$

$$\frac{12 \cdot 11}{2 \cdot 1} \qquad \frac{14}{1}$$

$$66 \cdot 14$$

$$= 924$$

Binomial Theorem

Let x and y be variables, and let n be a positive integer. Then

$$(x+y)^n = \sum_{j=0}^{n} C(n, j) x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} y^n$$

Pascal’s Triangle

C(0, 0)

C(1, 0) C(1, 1)

C(2, 0) C(2,1) C(2, 2)

C(3, 0) C(3, 1) C(3, 2) C(3,3)

C(4, 0) C(4, 1) C(4, 2) C(4,3) C(4, 4)

C(5, 0) C(5, 1) C(5, 2) C(5,3) C(5, 4) C(5, 5)

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

The Pigeonhole Principle

Pigeonhole principle

If n pigeon holes are occupied by n+1 or more pigeons, then at least one pigeon hole is occupied more than one pigeon.

Example:

Suppose a department contains 13 professors. Then two of the professors (pigeons) were born in the same month (pigeon holes).

The Pigeonhole Principle

<u>Generalized pigeonhole principle</u>

If n pigeonholes are occupied by kn+1 or more pigeons, where k is a positive integer, the at least one pigeon hole is occupied by k+1 or more pigeons.

Example:

Find the minimum number of students in a class to be sure that three of them are born in the same month.

Here the n=12 months are the pigeonholes and k+1=3, or k=2.Hence among any kn+1=25students (pigeons), three of them are born in the same month.